Europ. J. Combinatorics (1999) **20**, 713–724
Article No. eujc.1999.0245
Available online at http://www.idealibrary.com on IDEAL

Characterizing Combinatorial Geometries by Numerical Invariants

JOSEPH E. BONIN AND WILLIAM P. MILLER

We show that the projective geometry $PG(r-1, q)$ for $r > 3$ is the only rank-r (combinatorial) geometry with $(q^r - 1)/(q - 1)$ points in which all lines have at least $q + 1$ points. For $r = 3$, these numerical invariants do not distinguish between projective planes of the same order, but they do distinguish projective planes from other rank-3 geometries. We give similar characterizations of affine geometries. In the core of the paper, we investigate the extent to which partition lattices and, more generally, Dowling lattices are characterized by similar information about their flats of small rank. We apply our results to characterizations of affine geometries, partition lattices, and Dowling lattices by Tutte polynomials, and to matroid reconstruction. In particular, we show that any matroid with the same Tutte polynomial as a Dowling lattice is a Dowling lattice.

1. INTRODUCTION

There are several results that use basic counting information to characterize classes of simple matroids (combinatorial geometries; henceforth shortened to geometries). One result of this type is the following theorem of Greene [9], which extends earlier work of Basterfield and Kelly.

PROPOSITION 1.1. *Every geometry has at least as many copoints as points; equality holds if and only if the geometry is modular.*

Since projective geometries are the only connected modular geometries of rank 3 or more, it follows that any geometry of rank 3 or more with as many points as copoints and with no two-point lines is a projective geometry. If there are $(q^r - 1)/(q - 1)$ points (and copoints) and the rank r exceeds three, we can conclude that q is a prime power and the geometry is the unique rank-r projective geometry of order q, namely $PG(r-1, q)$, the geometry constructed from the field of order q.

We derive several results of this type. In Section 2 we show that any rank-r geometry with $(q^r - 1)/(q - 1)$ points in which all lines have at least $q + 1$ points is a projective geometry of order q. Since there can be many projective planes of a given order, this counting information distinguishes projective planes from other rank-3 geometries, but it does not generally characterize a given projective plane. Related results for affine geometries are also treated.

Dowling lattices have many properties similar to those of projective geometries, so it is natural to look for analogous results about Dowling lattices. We treat these in Section 3, which is the core of the paper.

In the last two sections, we apply our results to characterizations of affine geometries and Dowling lattices by Tutte polynomials, and to matroid reconstruction. Apart from relatively simple examples like projective geometries and uniform matroids, few classes of matroids are known to be characterized by their Tutte polynomials. We show that any matroid with the same Tutte polynomial as a Dowling lattice is a Dowling lattice.

We assume the reader is familiar with basic matroid theory. Our notation and terminology follow [13] with the following common additions. The *colines* of a rank-r matroid M are the flats of rank $r - 2$, and the *copoints* (hyperplanes) are the flats of rank $r - 1$. As is

0195–6698/99/080713 + 12 $30.00/0

justified by well-known cryptomorphisms, we use the notions of geometry and geometric lattice interchangeably.

2. Characterizations of Affine and Projective Geometries

In this section we give several characterizations of projective and affine geometries by numerical invariants. We use the following result from [11].

PROPOSITION 2.1. *Rank-r geometries having no $(q+2)$-point line minors have at most $(q^r-1)/(q-1)$ points. This upper bound is attained only by projective geometries of order q.*

THEOREM 2.2. *Let M be a rank-r geometry on S with $|S|=(q^r-1)/(q-1)$ in which all lines have at least $q+1$ points. Then M is a projective geometry of order q.*

PROOF. By Proposition 2.1, we need only show that M has no $(q+2)$-point line minors. Therefore by the Scum theorem of Higgs (see [13]), it suffices to show that there are $q+1$ copoints over each coline. To show this, we first prove that each rank-i flat has exactly $(q^i-1)/(q-1)$ points. To see this, let F and F' be flats with $F=F'\vee x$ for a point $x\in F-F'$. Counting the points on lines through x, each of which has at least q points besides x, shows that there are at least $1+q|F'|$ points in F. It follows that each rank-i flat has at least $(q^i-1)/(q-1)$ points. Similarly, if any rank-i flat F had more points, induction on a saturated chain from F to S would force more than $(q^r-1)/(q-1)$ points in S, contrary to the hypothesis. Thus each rank-i flat has precisely $(q^i-1)/(q-1)$ points. Thus there are

$$\frac{q^r-1}{q-1}-\frac{q^{r-2}-1}{q-1}=\frac{q^r-q^{r-2}}{q-1}$$

points outside each coline, and each copoint over the coline contains

$$\frac{q^{r-1}-1}{q-1}-\frac{q^{r-2}-1}{q-1}=\frac{q^{r-1}-q^{r-2}}{q-1}$$

of these points. Thus there are $(q^r-q^{r-2})/(q^{r-1}-q^{r-2})$, or $q+1$, copoints over each coline, as needed. □

We next present several characterizations of affine geometries. These are based on the following result from [2].

PROPOSITION 2.3. *Rank-r geometries having no $(q+2)$-point line minors and no $(q+1)$-point lines have at most q^{r-1} points. This upper bound is attained only by affine geometries of order q.*

THEOREM 2.4. *Assume M is a rank-r geometry with q^{r-1} points in which lines have q points and planes have at least q^2 points. Then M is an affine geometry of order q.*

Theorem 2.4, which plays a key role in Section 4, is the case $j=2$ of the following theorem.

THEOREM 2.5. *Let j be an integer with $2\le j\le r-2$. Assume M is a rank-r geometry with q^{r-1} points in which lines have at most q points, rank-$(j-1)$ flats have q^{j-2} points, rank-j flats have q^{j-1} points, and rank-$(j+1)$ flats have at least q^j points. Then M is an affine geometry of order q.*

PROOF. As above, we need only show that there are $q+1$ copoints over each coline. This follows by the same type of counting as at the end of the proof of Theorem 2.2 once we establish that for each $i \geq j$, each rank-i flat has exactly q^{i-1} points. To see that for $i \geq j$ each rank-i flat has at least q^{i-1} points, induct on i. Assume F' is a rank-i flat with at least q^{i-1} points and that F is a rank-$(i+1)$ flat with $F = F' \vee x$ for a point $x \in F - F'$. Fix a rank-$(j-1)$ flat Y in F'. Since $|Y| = q^{j-2}$ and all rank-j flats have q^{j-1} points, Y is in at least $(q^{i-1} - q^{j-2})/(q^{j-1} - q^{j-2})$ rank-j flats in F'. The point x from $F - F'$ together with any rank-j flat in F' containing Y determines a rank-$(j+1)$ flat in F. Only points in the rank-j flat $Y \vee x$ are in more than one such rank-$(j+1)$ flat and each such rank-$(j+1)$ flat has at least $q^j - q^{j-1}$ points in $F - (Y \vee x)$. Thus F has at least

$$q^{j-1} + \frac{q^{i-1} - q^{j-2}}{q^{j-1} - q^{j-2}}(q^j - q^{j-1}) = q^i$$

points. As in the last proof, equality follows since M has q^{r-1} points. □

We turn to another result of this type. The proof is valid only for q greater than two, although it seems likely that the statement is correct also for $q = 2$.

THEOREM 2.6. *Assume q exceeds 2 and M is a rank-r geometry with q^{r-1} points in which lines have q points and copoints have q^{r-2} points. Then M is an affine geometry of order q.*

PROOF. Again we need only show that there are at most $q+1$ copoints over each coline. Assume a coline C is covered by $q+2$ or more copoints. Since there are

$$(q^{r-1} - |C|)/(q^{r-2} - |C|)$$

copoints over C, we get $|C| \geq 2q^{r-2}/(q+1)$. Selecting a point $x \notin C$ in a hyperplane H over C and counting the points on lines through x and, in turn, the $2q^{r-2}/(q+1)$ or more points of C shows that there are at least $1 + (q-1)2q^{r-2}/(q+1)$ points in H. Since this exceeds q^{r-2}, this contradiction shows that no coline is covered by $q+2$ or more copoints, as needed. □

3. A CHARACTERIZATION OF DOWLING LATTICES

In this section we derive a result for Dowling lattices akin to Theorems 2.2 and 2.4. Because Dowling lattices lack analogs of Propositions 2.1 and 2.3, and the number of points in a flat is not constant for each rank, the argument is more involved than those in Section 2. We start by recalling Dowling lattices, which we present via line-closure. For a complete treatment from a different perspective, see Dowling [8].

Let M be a geometry on S. A *subset T of S is line-closed* if for every two points $x, y \in T$, the line $x \vee y$ is contained in T. A *geometry M is line-closed* if the flats of M are precisely the line-closed sets. Dowling lattices are supersolvable [8], and supersolvable geometries are line-closed [10]. Thus Dowling lattices can be defined by specifying their points and lines.

Let G be a finite group, written with multiplicative notation. The *rank-r Dowling lattice over G*, denoted $Q_r(G)$, has the following points and lines. There are two kinds of points: *joints* $p_1, p_2, \ldots, p_r$, which form a basis for $Q_r(G)$; and *internal points* g_{ij} for every $g \in G$ and every pair of indices with $1 \leq i < j \leq r$. Hence $Q_r(G)$ has $r + \binom{r}{2}|G|$ points. There are two types of nontrivial lines (i.e., lines with at least three points): *coordinate lines* $p_i \vee p_j = \{p_i, p_j\} \cup \{g_{ij} | g \in G\}$; and *transversal lines* $\{g_{ij}, h_{jk}, (gh)_{ik}\}$ for each pair

$g, h \in G$ and triple of indices with $1 \le i < j < k \le r$. Thus the transversal lines are contained in the *coordinate planes* $p_i \vee p_j \vee p_k$, and they encode the group operation. For $i > j$ we set $g_{ij} = (g^{-1})_{ji}$, where g^{-1} is the inverse of g in the group G. With this convention we can drop the restriction $i < j < k$ in the definition of transversal lines.

Dowling lattices can be defined for infinite groups in the same manner; however, our results concern only the finite case. For $r = 3$, G need only be a quasigroup for line-closure to give rise to a geometry on the points of $Q_3(G)$. If G is the trivial (one-element) group, then $Q_r(G)$ is isomorphic to the rank-r partition lattice Π_{r+1}. Thus Dowling lattices generalize partition lattices.

We use the following characterization of Dowling lattices from [3]. In [1] it is observed that when $r = 3$, the axioms below characterize Dowling lattices based on quasigroups.

PROPOSITION 3.1. *A geometry M of rank $r \ge 4$ is a Dowling lattice if and only if M has points $p_1, p_2, \ldots, p_r$ satisfying these axioms.*

(D1) *Each point of M lies on a coordinate line $p_i \vee p_j$.*
(D2) *No coordinate line $p_i \vee p_j$ is trivial.*
(D3) *For points $x \in (p_i \vee p_j) - \{p_i, p_j\}$ and $y \in (p_i \vee p_k) - \{p_i, p_k\}$, the line $x \vee y$ is nontrivial.*

Note that (D1) and the rank imply that $p_1, p_2, \ldots, p_r$ form a basis of M. In the case of a nontrivial group G, the basis $p_1, p_2, \ldots, p_r$ of $Q_r(G)$ satisfying (D1)–(D3) is unique. Since Π_{r+1} has $r+1$ such bases, special consideration is needed for Π_{r+1}; we treat this case first.

It is immediate that the counting conditions (1) to (4) in the next theorem hold in Π_{r+1}. We show that these statistics about flats of the first four ranks characterize Π_{r+1}.

THEOREM 3.2. *Assume that a rank-r geometry M has*

(1) $\binom{r+1}{2}$ *points,*
(2) $\binom{r+1}{3}$ *lines with three points,*
(3) *no five-point planes, $\binom{r+1}{4}$ planes with six points, no planes with more than six points, and*
(4) *no rank-4 flats with more than 10 points.*

Then $M \cong \Pi_{r+1}$.

PROOF. We first prove that M has several properties that hold for Π_{r+1}, namely:

(a) the number of three-point lines through each point x is exactly $r - 1$;
(b) the number of six-point planes through each point x is exactly $\binom{r-1}{2}$; and
(c) all six-point planes are isomorphic to Π_4.

Let $\ell_1, \ell_2, \ldots, \ell_t$ be the three-point lines through x. By (3), no three of these lines are coplanar. Therefore the three-point lines through x determine $\binom{t}{2}$ planes $\ell_i \vee \ell_j$ through x, each having at least five points. Hence by (3) these are six-point planes. Let $x_{ij} = x_{ji}$ be the unique point of $\ell_i \vee \ell_j$ on neither ℓ_i nor ℓ_j.

We claim that there are $\binom{t}{2}$ distinct points x_{ij}. Note that $x_{ij} \ne x_{ik}$ since $\ell_i \vee \ell_j$ and $\ell_i \vee \ell_k$ meet in the line ℓ_i. To see that $x_{ij} \ne x_{hk}$ when $|\{i, j, h, k\}| = 4$, note that if $x_{ij} = x_{hk}$, then the planes $\ell_i \vee \ell_j$ and $\ell_h \vee \ell_k$ intersect in the line $x \vee x_{ij}$. Therefore $\ell_i \vee \ell_j \vee \ell_h \vee \ell_k$ has rank 4. However $\ell_i \vee \ell_j \vee \ell_h \vee \ell_k$ contains at least 12 points, namely the nine points from $\ell_i, \ell_j, \ell_h, \ell_k$ together with x_{ij}, x_{ih} and x_{ik}. This contradiction of (4) establishes the claim.

The lines $\ell_1, \ell_2, \ldots, \ell_t$ through x and the points x_{ij} account for $2t + 1 + \binom{t}{2}$ points of M. Therefore $2t + 1 + \binom{t}{2} \leq \binom{r+1}{2}$, so $t \leq r - 1$. However, by (1) and (2), the average number of three-point lines through a point is $3\binom{r+1}{3}/\binom{r+1}{2} = r - 1$. Hence the number of three-point lines through each point of M is exactly $r - 1$, establishing (a).

To prove (b), note that while each point x lies in at least $\binom{r-1}{2}$ planes having six points (the planes $\ell_i \vee \ell_j$ discussed above), the average number of six-point planes through a given point is also $6\binom{r+1}{4}/\binom{r+1}{2} = \binom{r-1}{2}$.

We have established that there are $\binom{r+1}{2}\binom{r-1}{2}$ pairs of three-point lines intersecting in a point, and each such pair is in exactly one six-point plane. Since a point in a six-point plane can be on at most two three-point lines in that plane, the number of pairs of three-point lines intersecting in a point is at most $\binom{r+1}{4}6$, which is $\binom{r+1}{2}\binom{r-1}{2}$. Therefore each point in each six-point plane is on two three-point lines in that plane. From this, (c) follows.

We now select a basis for M satisfying axioms (D1)–(D3) of Proposition 3.1. Let x be any point of M, let $\ell_1, \ell_2, \ldots, \ell_{r-1}$ be the three-point lines through x, let $\ell_i = \{x, x_i, x_i'\}$, and let x_{ij} be the unique point in $\ell_i \vee \ell_j$ on neither ℓ_i nor ℓ_j. Define an equivalence relation $\sim$ on $\{x_1, x_1', x_2, x_2', \ldots, x_{r-1}, x_{r-1}'\}$ by: $z_1 \sim z_2$ if and only if either $z_1 = z_2$ or $z_1 \vee z_2$ is a three-point line not among the lines $\ell_1, \ell_2, \ldots, \ell_{r-1}$. To prove transitivity, assume, without loss of generality, that $x_i \sim x_j$ and $x_j \sim x_k$ with x_i, x_j, x_k distinct. The plane $x_i \vee x_j \vee x_k$ meets the plane $\ell_i \vee \ell_j$ in the three-point line $x_i \vee x_j$ and meets the plane $\ell_j \vee \ell_k$ in the three-point line $x_j \vee x_k$. Therefore $x_i \vee x_j \vee x_k$ is a six-point plane. Since (4) implies

$$\ell_i \vee \ell_j \vee \ell_k = (\ell_i \vee \ell_j) \cup (\ell_i \vee \ell_k) \cup (\ell_j \vee \ell_k),$$

we have $x_{ik} \in x_i \vee x_j \vee x_k$. Therefore x_{ik} is on the line $x_i \vee x_k = (x_i \vee x_j \vee x_k) \wedge (\ell_i \vee \ell_k)$. Thus $x_i \sim x_k$.

By (c) for each i and j, x_i (and similarly x_i') is on a three-point line with exactly one of x_j and x_j'. Therefore there are exactly two equivalence classes and each line ℓ_i contains one member of each class. Let $p_1, p_2, \ldots, p_{r-1}$ be one equivalence class and let $p_r = x$.

To see that $p_1, p_2, \ldots, p_r$ satisfy axioms (D1)–(D3) of Proposition 3.1, note first that by construction, each $p_i \vee p_j$ is a three-point line so (D2) holds. Axiom (D1) holds since the union of the lines $p_i \vee p_j$ contains $\binom{r}{2} + r = \binom{r+1}{2}$ points. Finally, (D3) holds since each $p_i \vee p_j \vee p_k$ is isomorphic to Π_4, and the lines $p_i \vee p_j$, $p_i \vee p_k$, and $p_j \vee p_k$ are three of the four three-point lines in Π_4. Hence M is the rank-r Dowling lattice $Q_r(G)$ with $|G| = 1$, so $M \cong \Pi_{r+1}$. □

To treat the corresponding result for Dowling lattices over nontrivial groups, we need the following result [8, Theorem 2].

PROPOSITION 3.3. *For each flat F in the Dowling lattice $Q_r(G)$, there are integers s and k with $0 \leq s \leq r$ and $k \geq 0$, and integers $n_1, n_2, \ldots, n_k$ with $n_i \geq 2$ such that $F \cong Q_s(G) \oplus \Pi_{n_1} \oplus \Pi_{n_2} \oplus \cdots \oplus \Pi_{n_k}$.*

This proposition and the following remarks underlie the counting that is needed for the proof of our characterization of Dowling lattices (Theorem 3.4). Assume $|G| > 1$. In $Q_r(G)$, flats F with $F \cong Q_s(G) \oplus \Pi_{n_1} \oplus \Pi_{n_2} \oplus \cdots \oplus \Pi_{n_k}$ have

$$|F| = s + \binom{s}{2}|G| + \sum_{i=1}^{k} \binom{n_i}{2}$$

points and rank

$$r(F) = s + \sum_{i=1}^{k}(n_i - 1).$$

For $s \geq 2$, such flats are formed in the following way. Let $p_1, p_2, \ldots, p_r$ be the joints of $Q_r(G)$ and let $\{t_1, t_2, \ldots, t_s\}$, $\{i_1, i_2, \ldots, i_{n_1}\}, \ldots, \{j_1, j_2, \ldots, j_{n_k}\}$ be pairwise disjoint subsets of $\{1, 2, \ldots, r\}$ with $\{t_1, t_2, \ldots, t_s\}$ serving as a 'distinguished' subset. For the distinguished subset, we form the flat $p_{t_1} \vee p_{t_2} \vee \cdots \vee p_{t_s}$, which is isomorphic to $Q_s(G)$. For the subset $\{i_1, i_2, \ldots, i_{n_1}\}$, choose one internal point x_h on each coordinate line $p_{i_1} \vee p_{i_h}$ for $2 \leq h \leq n_1$. The flat $x_2 \vee x_3 \vee \cdots \vee x_{n_1}$ is isomorphic to the partition lattice Π_{n_1}. (The points $x_2, x_3, \ldots, x_{n_1}$ serve as a basis for Π_{n_1} satisfying axioms (D1)–(D3) of Proposition 3.1.) Doing the same for the remaining subsets and considering the flat F spanned by all these points, we obtain $F \cong Q_s(G) \oplus \Pi_{n_1} \oplus \Pi_{n_2} \oplus \cdots \oplus \Pi_{n_k}$.

For every rank i with $2 \leq i \leq r-1$, there are at least two distinct cardinalities of flats of rank i in $Q_r(G)$; in particular, both $Q_i(G)$ and $Q_{i-1}(G) \oplus \Pi_2$ are among the isomorphism types for rank-i flats. For several ranks i, we will be concerned with the flats of rank i of the largest cardinality and those of the second largest cardinality. Those with the largest cardinality will be called *maximal rank-i flats*, while those of the second largest cardinality will be called *submaximal rank-i flats*. We will also use this terminology in the proof of Theorem 3.4 for those ranks about which we make assumptions on the cardinalities of flats. To motivate Theorem 3.4 and the counting used in the proof, we identify the maximal and submaximal flats in $Q_r(G)$ for the first several ranks. Since we have already dealt with $|G| = 1$ in Theorem 3.2, henceforth, we assume $|G| \geq 2$.

Lines in $Q_r(G)$ are of three types: $Q_2(G)$, $\Pi_2 \oplus \Pi_2$, and Π_3. These have $|G| + 2$, two and three points respectively. The maximal lines (the $(|G| + 2)$-point lines) are the $\binom{r}{2}$ coordinate lines $p_i \vee p_j$. The submaximal lines (the three-point lines) are the transversal lines. Since each transversal line lies in a coordinate plane $p_i \vee p_j \vee p_k$ and there are $|G|^2$ transversal lines in each such plane, the number of submaximal lines in $Q_r(G)$ is $\binom{r}{3}|G|^2$.

There are four types of planes with four or more points: $Q_3(G)$, $Q_2(G) \oplus \Pi_2$, $\Pi_3 \oplus \Pi_2$, and Π_4, having $3|G| + 3$, $|G| + 3$, four and six points respectively. Thus, the maximal planes are those isomorphic to $Q_3(G)$. Which planes are submaximal depends on $|G|$, but only maximal planes have $2|G| + 3$ or more points.

In rank 4, candidates for the maximal and submaximal flats are isomorphic to one of $Q_4(G)$, $Q_3(G) \oplus \Pi_2$, and Π_5, which have $6|G| + 4$, $3|G| + 4$, and 10 points respectively. Thus, the maximal rank-4 flats are isomorphic to $Q_4(G)$. The submaximal flats are isomorphic to $Q_3(G) \oplus \Pi_2$, unless $|G| = 2$ in which case the flats isomorphic to Π_5 are also submaximal.

For rank $i \geq 5$, the maximal flats are those isomorphic to $Q_i(G)$ (with $\binom{i}{2}|G| + i$ points), and the submaximal flats are those isomorphic to $Q_{i-1}(G) \oplus \Pi_2$ (with $\binom{i-1}{2}|G| + i$ points).

For our work in Section 4 it is important to note that the cardinality of submaximal flats of rank i exceeds that of maximal flats of rank $i - 1$. This holds since the maximal flats of rank $i - 1$ are isomorphic to $Q_{i-1}(G)$, and the flats isomorphic to $Q_{i-1}(G) \oplus \Pi_2$ have rank i but are not maximal.

The next theorem characterizes Dowling lattices using the cardinalities of the maximal rank-i flats for $i \leq 6$ and the cardinalities of the submaximal rank-i flats for $i = 2$ and 5. Since these statistics are shared by all Dowling lattices of a given rank based on groups of the same order, they do not determine a Dowling lattice uniquely unless the order of the group is prime. However, no geometries other than Dowling lattices can share these statistics about flats.

THEOREM 3.4. *Assume M is a rank-r geometry and $g > 1$ is an integer such that:*

(1) *M has $\binom{r}{2}g + r$ points,*

(2) *M has $\binom{r}{2}$ lines with $g + 2$ points, $\binom{r}{3}g^2$ lines with three points, and no other nontrivial lines,*

(3) *M has $\binom{r}{3}$ planes with $3g+3$ points, and no other planes with $2g+3$ or more points,*
(4) *M has $\binom{r}{4}$ rank-4 flats with $6g+4$ points, and no larger rank-4 flats,*
(5) *M has $\binom{r}{5}$ rank-5 flats with $10g+5$ points, $\binom{r}{4}\binom{r-4}{2}g$ rank-5 flats with $6g+5$ points, and no other rank-5 flats with more than $6g+4$ points,*
(6) *the rank-6 flats (if any) with most points have $15g+6$ points, and no other rank-6 flats have $14g+6$ or more points, and*
(7) *all rank-7 flats (if any) have fewer than $22g+8$ points.*

Then M is the Dowling lattice $Q_r(G)$ for some group (or quasigroup, if $r=3$) G of order g.

PROOF. If $r=3$, there are three maximal lines. Since there are $3g+3$ points, it follows that each pair of maximal lines intersects in a point, and there are exactly three such points of intersection, say p_1, p_2, p_3. Using the basis p_1, p_2, p_3 and the assumption that there are g^2 submaximal lines, it is easy to check that axioms (D1)–(D3) in Proposition 3.1 hold.

We treat ranks 4 and higher through a series of deductions about the structure of maximal flats, especially those of ranks 4 and 5.

DEDUCTION 3.5. *Three maximal planes in a rank-4 flat F cannot intersect in a line.*

PROOF. If three maximal planes in F intersect in a line containing m points, then together these planes contain $m+3(3g+3-m)$ points. Thus we have $9g+9-2m \le |F| \le 6g+4$. However, since $m \le g+2$, this is impossible. □

DEDUCTION 3.6. *No point of a maximal rank-4 flat F can be in more than three maximal planes of F*

PROOF. Consider a point x in three maximal planes P_1, P_2, P_3 of F. Let $\ell_{ij} = P_i \cap P_j$ and let $m_{ij} = |\ell_{ij} - \{x\}|$. Thus $m_{ij} \le g+1$. Note that $P_1 \cup P_2 \cup P_3$ is the disjoint union of $\{x\}$, the three sets $\ell_{ij} - \{x\}$, and the three sets $P_i - (\ell_{ij} \cup \ell_{ik})$. Thus $|P_1 \cup P_2 \cup P_3|$ is

$$1+m_{12}+m_{13}+m_{23}+(3g+2-m_{12}-m_{13})+(3g+2-m_{12}-m_{23})+(3g+2-m_{13}-m_{23}),$$

or $9g+7-m_{12}-m_{13}-m_{23}$. Since $|F| = 6g+4$ and $m_{ij} \le g+1$, we obtain $m_{ij} = g+1$, and so $P_1 \cup P_2 \cup P_3 = F$. Thus P_1, P_2, P_3 each contain two maximal lines through x and precisely g points not on these lines, and x is on precisely three maximal lines in F. Therefore x cannot be in a fourth maximal plane in F. □

DEDUCTION 3.7. *Each maximal rank-4 flat contains at most four maximal planes*

PROOF. Assume the maximal rank-4 flat F contains $i \ge 4$ maximal planes. Consider the set $\mathcal{P}$ of pairs (x, P) where P is a maximal plane in F and $x \in P$. Since there are i maximal planes, $|\mathcal{P}| = i(3g+3)$. Each point is in at most three maximal planes by 3.6. If each point were in at most two maximal planes, then $|\mathcal{P}| \le 2(6g+4)$, or $i(3g+3) \le 2(6g+4)$, which is impossible. Thus some point is in three maximal planes.

Let x be such a point. By the proof of 3.6, x is on three maximal lines, ℓ_1, ℓ_2, ℓ_3 in F, and the three maximal planes containing x are $\ell_1 \vee \ell_2, \ell_1 \vee \ell_3, \ell_2 \vee \ell_3$. Let P be a fourth maximal plane in F. By 3.5, P intersects each of the lines ℓ_1, ℓ_2, ℓ_3 in at most one point. Since there are $3g$ points of F not on these lines and $|P| = 3g+3$, it follows that P intersects each of these lines in a single point and consists of these three points of intersection and the $3g$ points of F not on these lines. Since $g \ge 2$, the $3g$ points of P not on ℓ_1, ℓ_2, ℓ_3 are not collinear and hence span P. Thus there is only one maximal plane not containing x. □

We can say more about maximal rank-4 flats containing four maximal planes.

DEDUCTION 3.8. *Let F be a maximal rank-4 flat containing four maximal planes P_1, P_2, P_3, and P_4. Then each of the four intersections $P_i \cap P_j \cap P_k$ is a single point, p_h, where $\{i, j, k, h\} = \{1, 2, 3, 4\}$. The maximal lines of F are the six lines $P_i \cap P_j = p_k \vee p_h$. The plane P_i is $p_j \vee p_k \vee p_h$. Each point of F is on a maximal line. Furthermore, each three-point line of F is in some plane P_i, and no three-point line contains any of the points p_1, p_2, p_3, p_4. Therefore F contains at most $4g^2$ submaximal lines.*

PROOF. From the work above, only the claims in the second to last sentence require proof. All points not on maximal lines with p_i are in P_i; thus any three-point line with p_i would lie in P_i, contrary to F having rank 4. Since the three points on a submaximal line must be on distinct lines $p_i \vee p_j$, it follows that two of the points are on $p_i \vee p_j$ and $p_i \vee p_k$ respectively for some i, j, and k. Thus two of the three points are in P_h, so the line is in P_h. (The same argument also shows that maximal lines lie in maximal planes, and hence there are indeed only six maximal lines.) □

If r is 4, by (2) there are precisely $4g^2$ submaximal lines. From Deduction 3.8 and Proposition 3.1, it follows that M is a Dowling lattice. Thus we turn to ranks 5 and greater.

It should cause no confusion to refer to the points p_1, p_2, p_3, p_4 in Deduction 3.8 as *joints*. We shall adopt the same terminology for the analogous distinguished points in higher-rank flats as the need arises.

DEDUCTION 3.9. *Each rank-5 flat contains at most five maximal rank-4 flats.*

PROOF. Let F be a rank-5 flat and let $T_1, T_2, \ldots, T_t$ be the maximal rank-4 flats in F. Since

$$|T_i \cup T_j| = |T_i| + |T_j| - |T_i \cap T_j| = 2(6g + 4) - |T_i \cap T_j| \leq |F| \leq 10g + 5,$$

we get $|T_i \cap T_j| \geq 2g + 3$. We conclude that $T_i \cap T_j$ is a maximal plane. By similar counting, we get that the maximal planes $T_1 \cap T_2, T_1 \cap T_3, \ldots, T_1 \cap T_t$ in the rank-4 flat T_1 are distinct. Thus $t \leq 5$ by 3.7. □

DEDUCTION 3.10. *Let F be a maximal rank-5 flat containing five maximal rank-4 flats. Then there are five points p_1, p_2, p_3, p_4, p_5 such that the 10 lines $p_i \vee p_j$ are precisely the maximal lines of F and these lines contain all points of F. The five maximal rank-4 flats are the flats $p_i \vee p_j \vee p_k \vee p_h$ for $\{i, j, k, h\} \subset \{1, 2, 3, 4, 5\}$. All submaximal lines in F lie in planes of the form $p_i \vee p_j \vee p_k$; hence there are at most $10g^2$ submaximal lines in F.*

PROOF. Let T_1, T_2, T_3, T_4, T_5 be the maximal rank-4 flats in F. Each T_i contains four maximal planes, namely, the intersections of T_i with the other four maximal rank-4 flats. Thus 3.8 applies to each T_i. Let p_2, p_3, p_4, p_5 be the joints of T_1 and we may assume that $T_1 \cap T_2$ is the maximal plane $p_3 \vee p_4 \vee p_5$. Thus there is a point p_1 in T_2 such that p_1, p_3, p_4, p_5 are the joints of T_2. By considering T_3, we deduce that $p_1 \vee p_2$ is also a maximal line. Counting shows that the union of the lines $p_i \vee p_j$ is F. Since each point of F is on one of the lines $p_i \vee p_j$ and two points suffice to span a line, it follows that all lines lie in maximal rank-4 flats. Thus our assertions about submaximal lines, as well as that the lines $p_i \vee p_j$ are the only maximal lines, follow from 3.8. □

If r is 5, by (2) there are precisely $10g^2$ submaximal lines, so by 3.10 and Proposition 3.1, M is a Dowling lattice. We now treat all higher ranks.

DEDUCTION 3.11. *Each maximal rank-4 flat is in exactly $r-4$ maximal rank-5 flats.*

PROOF. We first show that each maximal rank-4 flat is in at most $r-4$ maximal rank-5 flats. Let $F_1, F_2, \ldots, F_t$ be the maximal rank-5 flats containing the maximal rank-4 flat F. Since F_i and F_j cover $F_i \cap F_j = F$, the flat $F_{ij} = F_i \vee F_j$ has rank 6. The set $F_i \cup F_j$ contains $6g+4+2(4g+1) = 14g+6$ points. It follows that each F_{ij} is a maximal flat and there are precisely g points in $L_{ij} = F_{ij} - (F_i \cup F_j)$. The sets L_{ij} are pairwise disjoint. To see this, first note that since F_{ij} and F_{ik} are rank-6 flats meeting in the rank-5 flat F_i, it follows that $L_{ij} \cap L_{ik} = \emptyset$. Next, if $x \in L_{ij} \cap L_{hk}$, where $|\{i, j, h, k\}| = 4$, then the rank-6 flats F_{ij} and F_{hk} cover their intersection $F \vee x$, and so $F_{ij} \vee F_{hk}$ has rank 7. However, the set $F_i \cup F_j \cup F_h \cup F_k$ contains $(6g+4)+4(4g+1) = 22g+8$ points, which by (7) is contrary to $F_{ij} \vee F_{hk}$ having rank 7. Hence the sets L_{ij} are pairwise disjoint. Thus the union of F with the flats $F_1, F_2, \ldots, F_t$ and the $\binom{t}{2}$ sets L_{ij} contains $6g+4+t(4g+1)+\binom{t}{2}g$ points. Since this is at most $\binom{r}{2}g+r$, the number of points in M, we have $t \leq r-4$.

To prove equality, let the maximal rank-4 flats be $S_1, S_2, \ldots, S_{\binom{r}{4}}$, and let S_i be contained in m_i maximal rank-5 flats. We just showed that $m_i \leq r-4$ and we are claiming equality holds. By (5), any non-maximal rank-5 flat containing S_i and a point $x \notin S_i$ is a submaximal rank-5 flat, namely $S_i \cup \{x\}$. Thus S_i is in $\left(\binom{r}{2}g+r\right) - (6g+4) - m_i(4g+1)$ submaximal rank-5 flats. There are $\binom{r}{4}\binom{r-4}{2}g$ submaximal rank-5 flats by (5). Note that each submaximal rank-5 flat F contains at most one maximal rank-4 flat F' since the singleton $F - F'$ is an isthmus in $M|F$, and by the cardinalities of planes, the maximal rank-4 flat F' can contain no isthmuses of $M|F'$. Thus there are at most $\binom{r}{4}\binom{r-4}{2}g$ maximal rank-4 flats contained in submaximal rank-5 flats. Therefore

$$\sum_{i=1}^{\binom{r}{4}} \left(\binom{r}{2}g + r - (6g+4) - m_i(4g+1) \right) \leq \binom{r}{4}\binom{r-4}{2}g.$$

Combining this with $m_i \leq r-4$, we get

$$\binom{r}{4}\left(\binom{r}{2}g + r - (6g+4) - (r-4)(4g+1) \right) \leq \binom{r}{4}\binom{r-4}{2}g.$$

However since equality holds here, we get $m_i = r-4$, as claimed. □

Since each maximal rank-5 flat contains at most five maximal rank-4 flats, there are at most $5\binom{r}{5}$ pairs of incident maximal rank-4 and maximal rank-5 flats. We have just shown that the number of such pairs is exactly $\binom{r}{4}(r-4)$. The equality of these expressions gives the next claim.

DEDUCTION 3.12. *Each maximal rank-5 flat contains precisely five maximal rank-4 flats.*

Thus 3.10 applies to all maximal rank-5 flats. With this we can now prove that M is a Dowling lattice. Let F be a maximal rank-4 flat and let $F_5, F_6, \ldots, F_r$ be the $r-4$ maximal rank-5 flats containing F. Let p_1, p_2, p_3, p_4 be the joints of F. By 3.10 for each i with $5 \leq i \leq r$, there is a point p_i such that p_1, p_2, p_3, p_4, p_i are the joints of F_i. By considering the rank-5 flat $p_1 \vee p_2 \vee p_3 \vee p_i \vee p_j$ where $5 \leq i < j \leq r$, which is maximal since it has at least $9g+5$ points, we deduce that $p_i \vee p_j$ is a maximal line. Since each plane $p_i \vee p_j \vee p_k$ with $1 \leq i < j < k \leq r$ contains three maximal lines, it is a maximal plane. Likewise each $p_i \vee p_j \vee p_k \vee p_h$ with $1 \leq i < j < k < h \leq r$ is a maximal rank-4 flat. Therefore the lines $p_i \vee p_j$ can intersect only in joints. In particular, the union of all lines $p_i \vee p_j$ contains $\binom{r}{2}g+r$ points. Thus, every point of M is on some maximal line $p_i \vee p_j$. It follows that each submaximal line is in a maximal plane $p_i \vee p_j \vee p_k$. Since each maximal plane contains at most g^2 submaximal lines, and there are $\binom{r}{3}g^2$ submaximal lines, each maximal plane contains exactly g^2 submaximal lines. From these conclusions, it is immediate that axioms (D1)–(D3) of Proposition 3.1 hold, proving that M is a Dowling lattice. □

4. Tutte Polynomials

In this section we apply the results in Sections 2 and 3 to investigate the extent to which affine geometries and Dowling lattices are characterized by their Tutte polynomials. We recall the essential background on Tutte polynomials; for more information, see [5, 7].

The *Tutte polynomial* is defined for a matroid M on the set S by

$$t(M; x, y) = \sum_{X : X \subseteq S} (x-1)^{r(M)-r(X)}(y-1)^{|X|-r(X)}.$$

From $t(M; x, y)$, one knows much about M, including the number of points, the rank, and whether M is connected. Certain other data, such as the number of copoints, cannot, in general, be determined from the Tutte polynomial (see Example 4.5 in [5]). While non-isomorphic matroids may have the same Tutte polynomial, certain matroids M have the property that M is the only matroid with Tutte polynomial $t(M; x, y)$. For instance, if $t(M; x, y) = t(PG(r-1, q); x, y)$, then M is a projective geometry of rank r and order q, so if $r > 3$, then M is isomorphic to $PG(r-1, q)$. This follows from results on perfect matroid designs in the fifth section of [5]. Alternatively one can argue that if $t(M; x, y) = t(PG(r-1, q); x, y)$, then M is a geometry, all lines in M have $q+1$ points, and M has $(q^r - 1)/(q-1)$ points, so Theorem 2.2 applies.

The *characteristic polynomial*, which plays a prominent role in many enumerative questions (see [7]), is related to the Tutte polynomial by

$$\chi(M; x) = (-1)^{r(M)} t(M; 1-x, 0).$$

The Tutte polynomial can be expressed in terms of characteristic polynomials via the *weighted characteristic polynomial* of a matroid M:

$$\overline{\chi}(M; x, y) = \sum_{X} x^{|X|} \chi(M/X; y)$$

where M/X is the contraction of M by the flat X (see [4]). (Note that the sum could also be taken over all sets X, rather than just flats X, since the characteristic polynomial of a matroid with loops is zero.) In terms of the weighted characteristic polynomial, the Tutte polynomial is given by

$$t(M; x, y) = \frac{1}{(y-1)^{r(M)}} \overline{\chi}(M; y, (x-1)(y-1)).$$

Thus for a matroid M on the set S, if one knows the characteristic polynomial of each upper interval $[X, S]$ in the lattice of flats as well as the cardinality of X, then one can compute the Tutte polynomial of M.

Dowling [8] proved that the characteristic polynomial of $Q_r(G)$ is given by

$$\chi(Q_r(G); x) = \prod_{i=0}^{r-1} (x - i|G| - 1).$$

Thus, the characteristic polynomial of $Q_r(G)$ depends only upon the rank r and the order $|G|$ of G. Dowling [8, Theorem 2] showed that the contraction $Q_r(G)/X$ of $Q_r(G)$ by a flat X

is isomorphic to $Q_i(G)$, where i is the rank of $Q_r(G)/X$. It follows that the characteristic polynomial of the contraction $Q_r(G)/X$ by a flat X depends only upon $|G|$, r, and the rank of X. From the description of the flats in [8], it is immediate that the number of flats of each rank i and cardinality j also depends only upon $|G|$, r, i, and j. Combining these results, we get the following proposition.

PROPOSITION 4.1. *If $|G|=|G'|$, then $\overline{\chi}(Q_r(G); x, y)=\overline{\chi}(Q_r(G'); x, y)$, hence*

$$t(Q_r(G); x, y) = t(Q_r(G'); x, y).$$

A considerably stronger form of the following result appears as Proposition 5.9 in [5] (in particular, see the discussion beginning on p. 195 of [5]).

PROPOSITION 4.2. *For a rank-r matroid M and any integer i with $0 \leq i \leq r$, let c_i be the largest cardinality among rank-i flats of M. Then for each i with $1 \leq i \leq r$ and each j with $c_{i-1} < j \leq c_i$, we can express the number of flats of M having rank i and cardinality j as a linear combination of the coefficients of the Tutte polynomial.*

Thus, the validity of all hypotheses in Theorems 2.4, 3.2, and 3.4 can be deduced from the Tutte polynomials $t(AG(r-1, q); x, y)$, $t(\Pi_{r+1}; x, y)$, and $t(Q_r(G); x, y)$ respectively. This gives the following corollary.

COROLLARY 4.3. *If $t(M; x, y) = t(AG(r-1, q); x, y)$, then M is an affine geometry of rank r and order q. Thus, if $r > 3$, then M is isomorphic to $AG(r-1, q)$.*

If $t(M; x, y) = t(\Pi_{r+1}; x, y)$, then M is isomorphic to Π_{r+1}.

If $t(M; x, y) = t(Q_r(G); x, y)$, then M is a Dowling lattice $Q_r(G')$ for some group (or quasigroup, if $r = 3$) G' of order $|G|$. Thus, if $|G|$ is a prime p and $r > 3$, then M is isomorphic to $Q_r(Z_p)$ where Z_p is the cyclic group of order p.

Proposition 4.1 shows that no more can be said about G' in the third case.

5. MATROID RECONSTRUCTION

There are several matroid problems analogous to the graph reconstruction problems (see [6, 12] and the references given there). We are concerned with reconstruction from hyperplanes. The *deck of hyperplanes* of a matroid M is the multiset of its unlabeled hyperplanes. That is, for each isomorphism type H of rank $r(M) - 1$, we know how many hyperplanes of M are isomorphic to H. A matroid M is *hyperplane reconstructible* if any matroid with the same deck of hyperplanes as M is isomorphic to M.

It is immediate that projective geometries of rank greater than three are hyperplane reconstructible since from the deck of hyperplanes, we can deduce the number of points and the number of copoints, and that there are no trivial lines. Projective planes of order q are hyperplane reconstructible if and only if there is a unique projective plane of order q.

Brylawski [6] has shown that the Tutte polynomial of a matroid can be reconstructed from the deck of hyperplanes. From this and Corollary 4.3, we get the following corollary.

COROLLARY 5.1. *If $r > 3$, then $AG(r-1, q)$ and $Q_r(G)$ are hyperplane reconstructible. The partition lattice Π_{r+1} is hyperplane reconstructible for all ranks $r > 1$.*

PROOF. The results for affine geometries and partition lattices are clear. Let M be a matroid of rank greater than three with the same deck of hyperplanes as $Q_r(G)$. From Brylawski's result, we know the Tutte polynomial of M, and from Corollary 4.3 we therefore know that M is a Dowling lattice. Since for $r > 3$, the only Dowling lattice having a hyperplane isomorphic to $Q_{r-1}(G)$ is $Q_r(G)$, the result follows. □

Acknowledgement

Theorem 2.6 was suggested by one of the referees; it weakens the hypotheses of a theorem from the original version of the paper.

References

1. M. K. Bennett, K. P. Bogart and J. Bonin, The geometry of Dowling lattices, *Adv. Math.*, **103** (1994), 131–161.
2. J. Bonin, Matroids with no $(q+2)$-point line minors, *Adv. Appl. Math.*, **17** (1996), 460–476.
3. J. Bonin and K. P. Bogart, A geometric characterization of Dowling lattices, *J. Comb. Theory, Ser. A*, **56** (1991), 195–202.
4. T. H. Brylawski, Intersection theory for embeddings of matroids into uniform geometries, *Stud. Appl. Math.*, **61** (1979), 211–244.
5. T. H. Brylawski, The Tutte polynomial. Part I: general theory, in: *Matroid Theory and Its Applications*, C.I.M.E., A. Barlotti (ed.), Liguori, Naples, 1980, pp. 125–175.
6. T. H. Brylawski, Hyperplane reconstruction of the Tutte polynomial of a geometric lattice, *Discrete Math.*, **35** (1981), 25–38.
7. T. H. Brylawski and J. G. Oxley, The Tutte polynomial and its applications, in: *Matroid Applications*, N. White (ed.), Cambridge University Press, Cambridge, 1992, pp. 123–225.
8. T. A. Dowling, A class of geometric lattices based on finite groups, *J. Comb. Theory, Ser. B*, **14** (1973), 61–86.
9. C. Greene, A rank inequality for finite geometric lattices, *J. Comb. Theory*, **9** (1970), 357–364.
10. M. D. Halsey, Line-closed combinatorial geometries, *Discrete Math.*, **65** (1987), 245–248.
11. J. P. S. Kung, Extremal matroid theory, in: *Graph Structure Theory*, N. Robertson and P. D. Seymour (eds), American Mathematical Society, Providence, RI, 1993, pp. 21–61.
12. W. P. Miller, Techniques in matroid reconstruction, *Discrete Math.*, **170** (1997), 173–183.
13. J. G. Oxley, *Matroid Theory*, Oxford University Press, Oxford, 1992.

Received 15 December 1996 in revised form 17 July 1998

JOSEPH E. BONIN AND WILLIAM P. MILLER
Department of Mathematics,
The George Washington University,
Washington DC, 20052,
U.S.A.
E-mail: jbonin@math.gwu.edu; wpm@math.gwu.edu
AND
Current address for W. P. Miller:
Department of Mathematics,
Computer Science, and Physics,
Rockhurst College,
Kansas City, MO 64110,
U.S.A.